# Simulation of reflected Brownian motion on two dimensional wedges and reflection principle

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# Abstract

- Two dimensional Brownian motion in a wedge : reflected and stopped
- Give density formulas
- (Exact) simulation algorithms
- Complexity and approximations

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# The reflection principle in one dimension

- $(W_t)$ : standard Brownian motion in one dimension
- $(X_t)$ : reflection of $(W_t)$ with respect to 0
- $T > 0$ : finite horizon

Then :

$$\mathbb{P}^x(W_T \in dy,\ \tau_0 > T) = \mathbb{P}^x(W_T \in dy, W_T > 0)\left(1 - e^{-\frac{x(x+y)}{T}}\right).$$

$$\mathbb{P}^x(X_T \in dy) = \mathbb{P}^x(W_T \in dy, W_T > 0)\left(1 + e^{-\frac{x(x+y)}{T}}\right).$$

- We can directly simulate $W_T \mathbb{1}_{\tau_0 > T}$ and $X_T$
- Symmetry in the formula between killed and reflected cases
- **Interpretation :** We substract (resp. add) the trajectories such that $W_T = y$ (resp. $X_T = y$) but such that $\tau_0 < T$.

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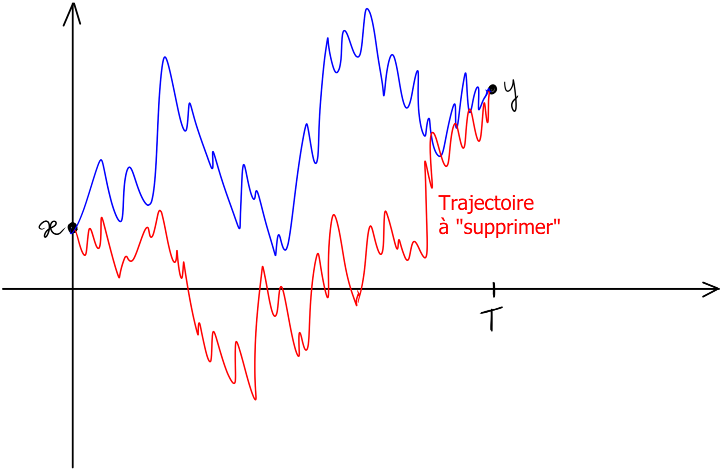


## Setting of the problem

- Two dimensional wedge $\mathcal{D}$ of angle $\alpha \in (0,\pi)$
- $(W_t)$ : two dimensional Brownian motion starting at $x_0 \in \mathcal{D}$
- $\tau = \inf\{t > 0,\ W_t \notin \mathcal{D}\}$
- $(X_t)$ : the reflection of $(W_t)$ on the wedge $\mathcal{D}$.

**Remarks :**

- Up to a rotation, we can assume that $(W_t)$ is non-correlated
- Varadhan, Williams, *Brownian motion in a wedge with oblique reflection*, 1985
  $\Longrightarrow$ Existence of the reflected process

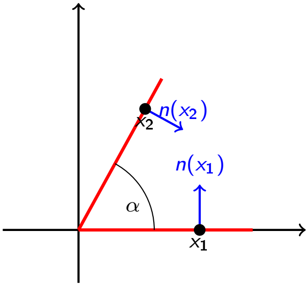
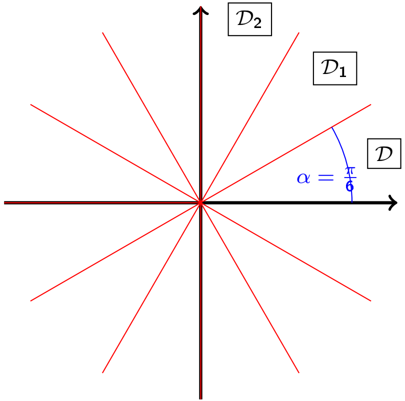



Figure – Example of wedge

# A two dimensional reflection principle

- Iyengar, *Hitting lines with two-dimensional Brownian motion*, 1985
- Assume that $\alpha = \frac{\pi}{m}$ for some $m \in \mathbb{N}^{\star}$, and define $T_k$ for $k = 0, ..., 2m-1$ :

$$T_k((r\cos\theta, r\sin\theta)) := (r\cos(\theta_k), r\sin(\theta_k))$$
$$\theta_k := \begin{cases} (k+1)\alpha - \theta; & k \text{ odd}, \\ k\alpha + \theta; & k \text{ even}. \end{cases}$$




$T_k : \mathcal{D} \to \mathcal{D}_k$

Figure – Partition of $\mathbb{R}^2$ using $\{\mathcal{D}_k\}_k$

### Density formulas for $\alpha = \frac{\pi}{m}$

$$\forall t > 0,\ x, y \in \mathcal{D},\ \mathbb{P}^x(X_t \in dy) = \frac{1}{2\pi t} \sum_{k=0}^{2m-1} e^{-\frac{|x - T_k y|^2}{2t}} dy. \quad (1)$$

$$\forall t > 0,\ x, y \in \mathcal{D},\ \mathbb{P}^x(W_t \in dy,\ \tau > t) = \frac{1}{2\pi t} \sum_{k=0}^{2m-1} (-1)^k e^{-\frac{|x - T_k y|^2}{2t}} dy \quad \text{(Iyengar)} \quad (2)$$

*Proof :* let $f : \mathcal{D} \to \mathbb{R}^+$ be a test function, and let

$$u(t, x) := \mathbb{E}\left[f(X_t)\right].$$

Then $u$ satisfies the partial differential equation with boundary conditions :

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathcal{D} \\ u(0,x) = f(x), & x \in \mathcal{D} \\ \nabla u(t,x) \cdot n(x) = 0, & x \in \partial\mathcal{D}. \end{cases} \quad (3)$$

## Density formulas for general $\alpha$

With $x = (r_0 \cos(\theta_0), r_0 \sin(\theta_0))$ and $y = (r\cos(\theta), r\sin(\theta))$, we have :

$$\mathbb{P}^x(X_t \in dy) = \frac{2r}{t\alpha} e^{-(r^2+r_0^2)/2t} \left( \frac{1}{2} I_0\left(\frac{rr_0}{t}\right) + \sum_{n=1}^{\infty} I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right) \cos\left(\frac{n\pi\theta}{\alpha}\right) \cos\left(\frac{n\pi\theta_0}{\alpha}\right)\right) dr d\theta. \tag{4}$$

$$\mathbb{P}^x(W_t \in dy,\ \tau > t) = \frac{2r}{t\alpha} e^{-(r^2+r_0^2)/2t} \sum_{n=1}^{\infty} I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right) \sin\left(\frac{n\pi\theta}{\alpha}\right) \sin\left(\frac{n\pi\theta_0}{\alpha}\right) dr d\theta. \tag{5}$$

**Bessel function :** $I_n$ is the modified Bessel function of order $n$, solution of the equation :

$$x^2 I_n''(x) + x I_n'(x) - (x^2 + n^2) I(x) = 0.$$

*Proof :* We write the formula for $\alpha = \frac{\pi}{m}$ in polar coordinates :

$$\mathbb{P}^x(X_t \in dy) = \frac{r}{2\pi t} e^{-(r^2+r_0^2)/2t} \sum_{k=0}^{2m-1} e^{(rr_0/t)\cos(\theta_0-\theta_k)} dr\ d\theta$$

And use the formulas

$$e^{\gamma z} = I_0(z) + 2\sum_{n=1}^{\infty} T_n(\gamma) I_n(z),$$

where $T_n$ is the Tchebychev polynomial of order $n$, and

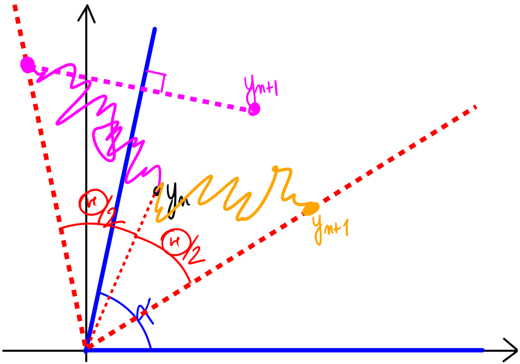
$$\sum_{k=0}^{2m-1} \cos(n(\theta_0 - \theta_k)) = \begin{cases} 2m\cos(n\theta)\cos(n\theta_0) & \text{if } n \text{ is a multiple of } m \\ 0 & \text{otherwise.} \end{cases}$$

More rigorously, one need to check that with $u(t,x) = \mathbb{E}[f(X_t)]$, $u$ satisfies :

$$\begin{array}{ll} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathcal{D} \\ u(0,x) = f(x), & x \in \mathcal{D} \\ \nabla u(t,x) \cdot n(x) = 0, & x \in \partial\mathcal{D}. \end{array} \tag{6}$$

# Simulation algorithm

- We use the box method, where the box is a wedge of angle $\Theta = \frac{\pi}{m} \leq \alpha$.
- Starting from $y_n$, we simulate the exit point of the wedge of angle $\Theta$ and centered on $y_n$.
- If the simulated point is outside the domain $\mathcal{D}$, then we reflect it with respect to the border of $\mathcal{D}$.



- We need to simulate the stopping point **and** the stopping time.
- We first simulate on which barrier $\pm$ we arrive
- Metzler, *Multivariate First-Passage Models in Credit Risk*, 2008 : We simulate the radius $r_\tau$ as :

$$r_\tau = \begin{cases} r_0 \left( \cos\left(m\theta_0\right) - \frac{\sin(m\theta_0)}{\tan((\pi - m\theta_0)(U-1))} \right)^{1/m} & \text{if } W_\tau \in \mathcal{V}^-, \\ r_0 \left( -\cos\left(m\theta_0\right) - \frac{\sin(m\theta_0)}{\tan(m\theta_0(U-1))} \right)^{1/m} & \text{if } W_\tau \in \mathcal{V}^+, \end{cases} \quad (7)$$

where $U \sim \mathcal{U}([0,1])$.

- Metzler, *On the first passage problem for correlated Brownian motion*, 2010

**Formula for the stopping time and the final point**

$$\mathbb{P}^x(\tau \in dt,\ W_\tau \in dy^{\pm}) = \frac{1}{2}\frac{\partial}{\partial n^{\pm}}\mathbb{P}^x(W_\tau \in dy,\ \tau > t)$$
$$= \frac{r_0}{2\pi t^2} e^{-\frac{r^2+r_0^2}{2t}} \sum_{k=0}^{m-1} \sin\left(\gamma_k^{\pm}\right) e^{\frac{rr_0}{t}\cos\left(\gamma_k^{\pm}\right)} drdt,$$

with $\gamma^+ = \alpha + \frac{2k\pi}{m} - \theta_0$ and $\gamma^- = \theta_0 - \frac{2k\pi}{m}$.

- This gives a formula to simulate according to the joint law of $(W_\tau, \tau)$.
- Problem : This is not a true mixture, as $\sin(\gamma^{\pm})$ can be negative. We use acceptance-rejection method.

# Complexity of the algorithm

- The number of iterations is the number of times the angle of a Brownian motion in $\mathbb{R}^2$ goes from $\frac{j\pi}{2m}$ to $\frac{(j\pm 1)\pi}{2m}$, i.e.

$$\tilde{\tau}_{i+1} := \inf \left\{ t > 0,\ |\theta(B_{t+\tilde{\tau}_i}) - \theta(B_{\tilde{\tau}_i})| \geq \frac{\pi}{2m} \right\}$$

$$N \sim \inf\{n \in \mathbb{N}^\star,\ \tilde{\tau}_1 + \dots + \tilde{\tau}_n > T\}.$$

- We want to study the process $\theta(B_t)_t$, but it is not Markovian.
- We use the skew-product representation :

**Skew-product representation**

$$B_t = R_t U_{F(t)}, \quad \text{with} \begin{cases} dR_t = d\hat{B}_t + \frac{1}{2}\frac{dt}{R_t} & \text{(Bessel)} \\ F(t) = \int_0^t \frac{ds}{R_s^2} \\ (U_t) \text{ is a Brownian motion on } \mathbb{S}^1 \end{cases}$$

**Proposition :** We have $\mathbb{E}[N] = +\infty$.
*Proof :* Denote $s_i$ the successive stopping times that $(\theta(U)_t)$ goes from $\frac{j\pi}{2m}$ to $\frac{(j\pm 1)\pi}{2m}$.
Then $s_i$ are i.i.d. and $\sum_{i=1}^{K} s_i = F\left(\sum_{i=1}^{K} \tilde{\tau}_i\right)$. Then

$$\mathbb{E}[N] = \sum_{K=1}^{\infty} \mathbb{P}(N \geq K) = \sum_{K=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{K} \tilde{\tau}_i \leq T\right)$$
$$= \sum_{K=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{K} s_i \leq F(T)\right) = \sum_{K=1}^{\infty} \int_0^{\infty} \mathbb{P}\left(\sum_{i=1}^{K} s_i \leq y\right) \mathbb{P}(F(T) \in dy)$$

But $\mathbb{E}[F(T)] = \mathbb{E}\left[\int_0^T \frac{ds}{R_s^2}\right] = \infty$, so $\mathbb{E}[N] = \infty$.

# Approximation algorithm

- If, during the simulation algorithm, $\frac{r_n^2}{T-T_n} < \epsilon$, then we approximate the distribution by taking only the first term, and immediately terminates the algorithm :

$$\mathbb{P}^{x_n}(X_T \in dy) \approx \frac{r}{t\alpha} e^{-\frac{r^2+r_n^2}{2(T-T_n)}} I_0\left(\frac{rr_n}{T-T_n}\right) drd\theta.$$

- **Proposition** : For all $p \in (1,2)$, we have

$$\mathbb{E}[N] \leq \frac{C(p,T,m)}{\epsilon^{p-1}}$$

- **Proposition** : We have $d_{TV}(\bar{X}_T, X_T) = O(\epsilon)$.

# Algorithms for general processes

- We consider a diffusion process $(Y_t)$ reflected in a wedge $\mathcal{D}$.
- We approximate $(Y_t)$ by its Euler-Maruyama scheme, giving a reflected Brownian motion with drift :

$$\bar{Y}_t = \bar{Y}_{t_k} + tb(\bar{Y}_{t_k}, t_k) + \sigma(\bar{Y}_{t_k}, t_k) \cdot B_t \quad \text{for } t \in [t_k, t_{k+1}].$$

- We apply the algorithm to simulate $\bar{Y}_{t_{k+1}}$ ; the drift can be dealt using a Girsanov change of measure.

- Parameters : $\alpha = 0.9$, $r_0 = 1$, $\theta_0 = 0.3$, and $f(r\cos(\theta), r\sin(\theta)) = r^2$.

| | $\mathbb{E}[f(W_\tau)]$ | 95 % interval | Time (s) | MC iterations |
|---|---|---|---|---|
| Metzler'algorithm | 1.515 | $\pm$ 0.074 | 4.40 | 10000 |
| This paper | 1.783 | $\pm$ 0.074 | 5.12 | 20000 |

Table – Estimation of $\mathbb{E}[f(W_\tau)]$

- With $T = 0.5$ :

| | $\mathbb{E}[f(W_{\tau\wedge T})]$ | 95 % interval | Time(s) | MC iterations |
|---|---|---|---|---|
| This paper | 1.409 | $\pm$ 0.057 | 1.69 | 2000 |

Table – Estimation of $\mathbb{E}[f(W_{\tau\wedge T})]$

| | $\mathbb{E}[f(X_T)]$ | 95 % interval | Time (s) | MC iterations |
|---|---|---|---|---|
| Reflected algorithm | 1.950 | $\pm$ 0.323 | 3.55 | 100 |
| Approximation with $\epsilon = 0.02$ | 2.135 | $\pm$ 0.082 | 6.63 | 2000 |

Table – Estimation of $\mathbb{E}[f(X_T)]$